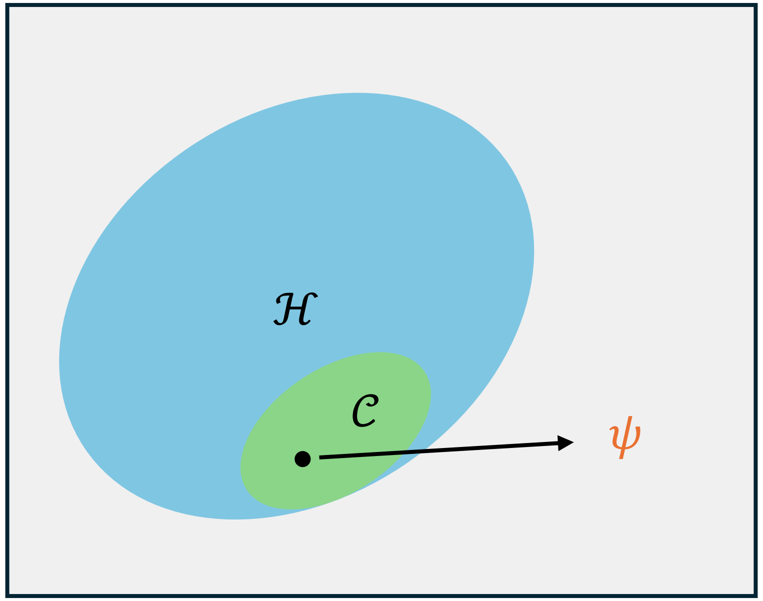
Nearly tight bounds for testing tree tensor network states

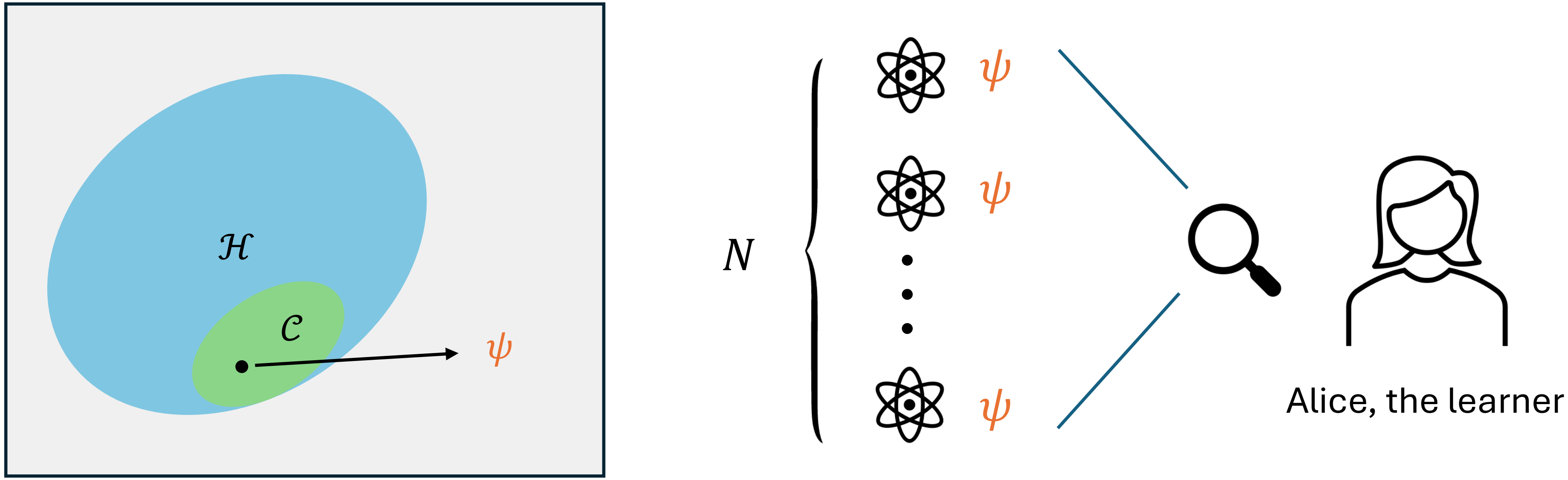
Benjamin Lovitz (Northeastern) and **Angus Lowe** (MIT)

arxiv:2410.21417

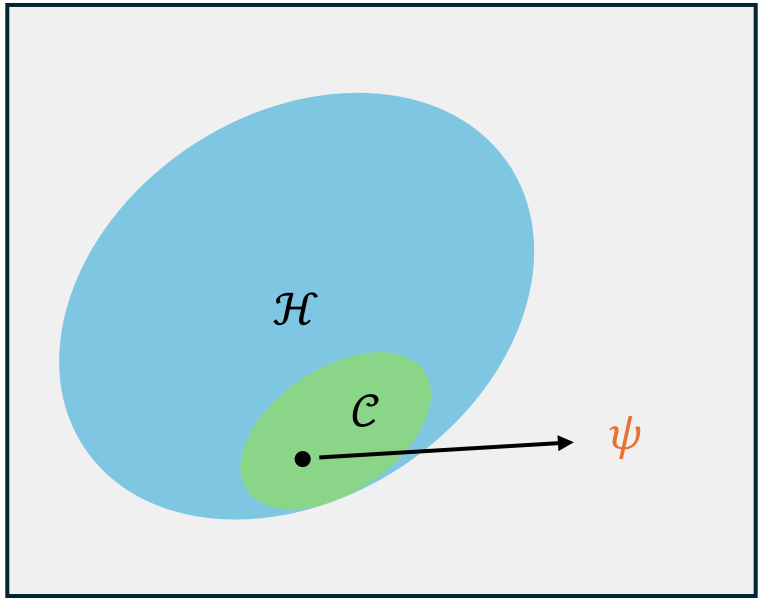
I. Background & Main Result
II. Lower Bound for a Specific Algorithm
III. There is No Better Algorithm
IV. Final Remarks

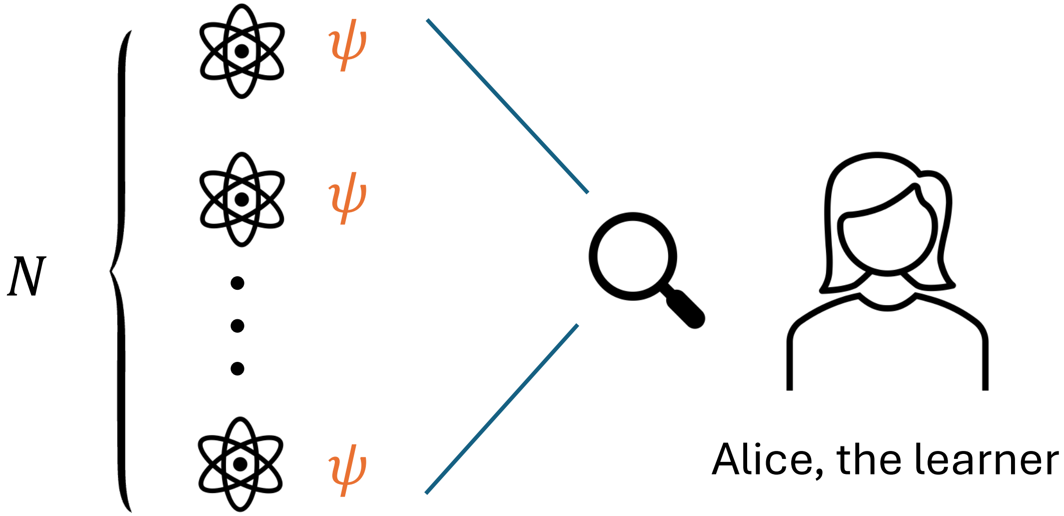


$\mathcal{H}$
$\mathcal{C}$
ψ

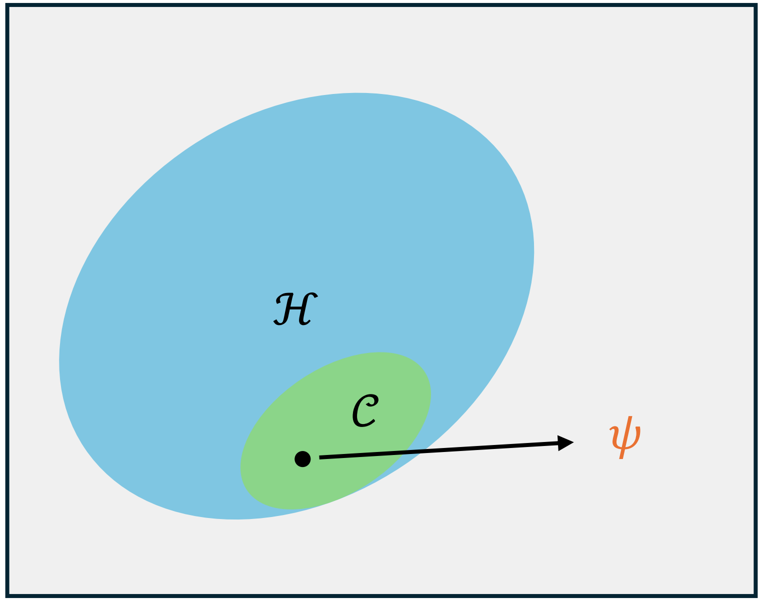


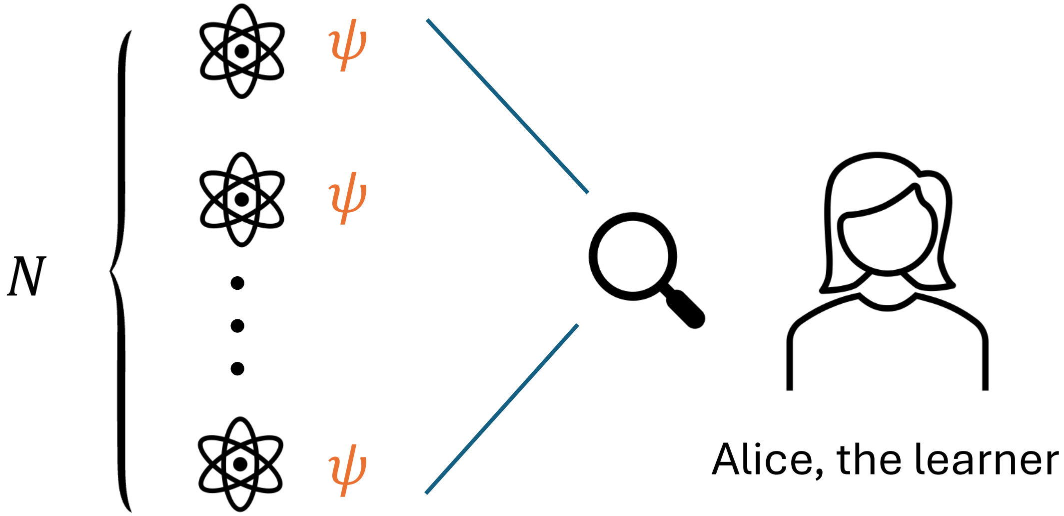
$\mathcal{H}$
$\mathcal{C}$
ψ
N
ψ
ψ
ψ
Alice, the learner





- What are the expected values of some observables? (Shadow tomography)
- How strong is the magnetic field that produces the state ψ? (Quantum sensing)
- What is ψ? (Tomography/learning)
- Is ψ even in $\mathcal{C}$? (Property testing)





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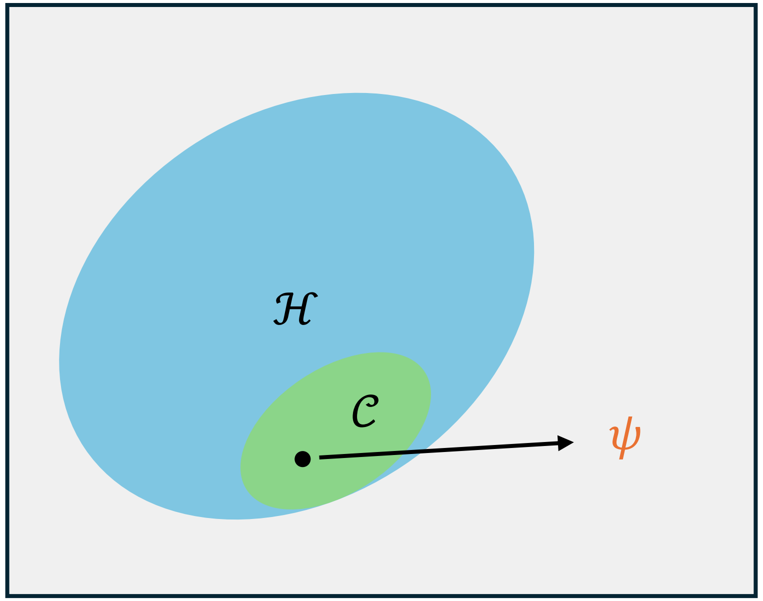

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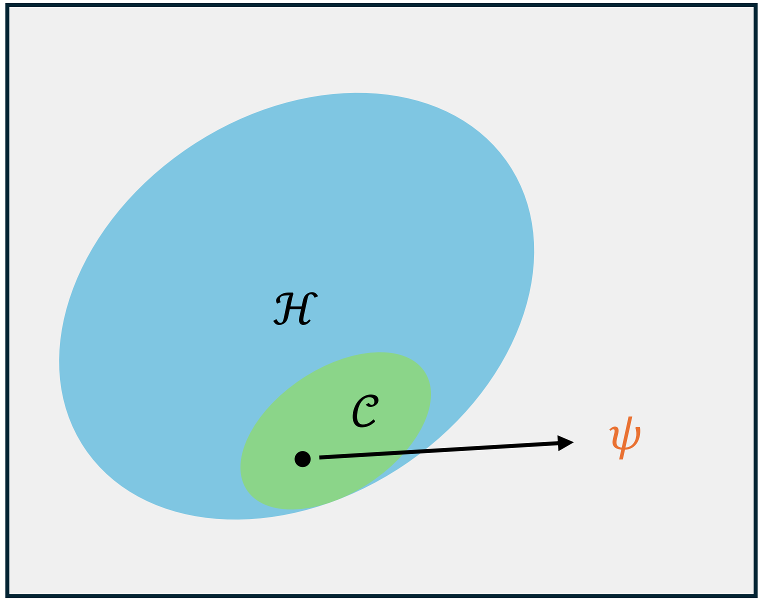

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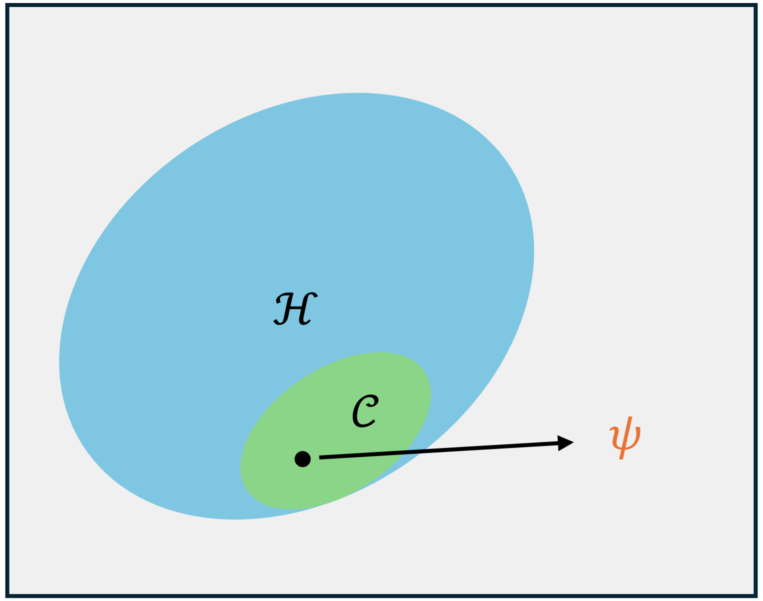

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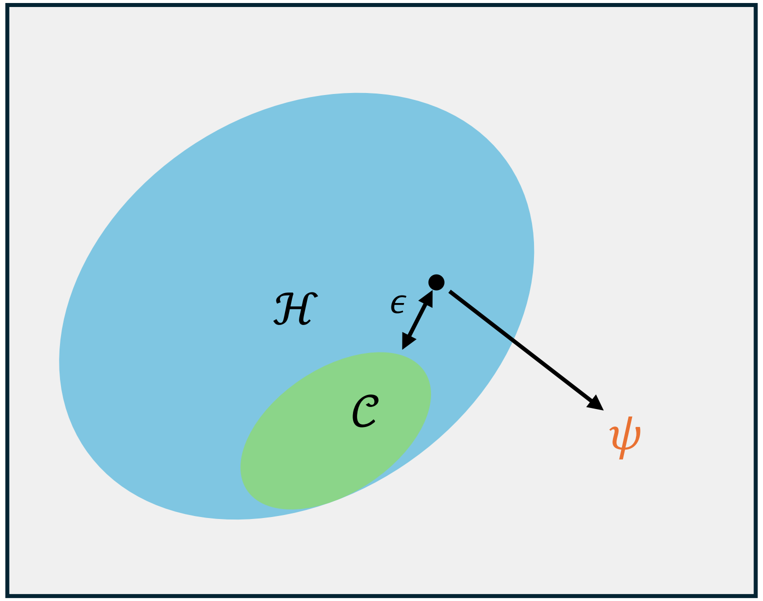

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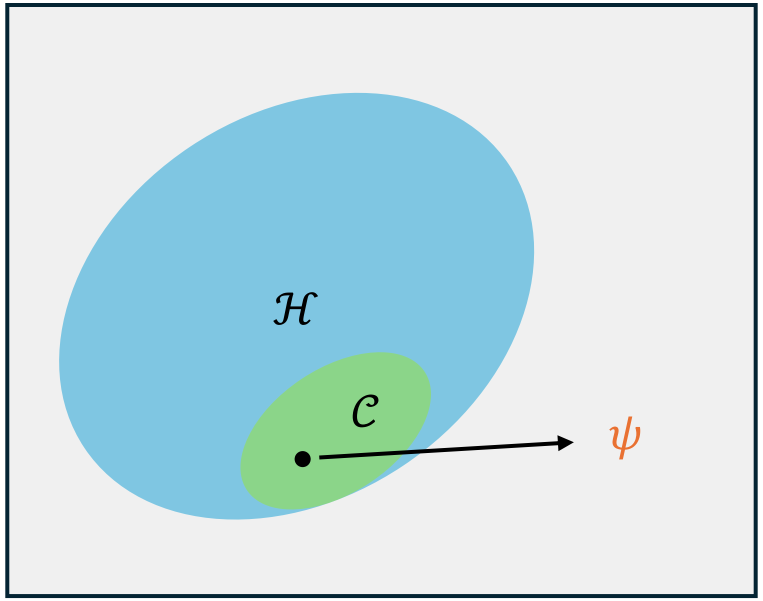

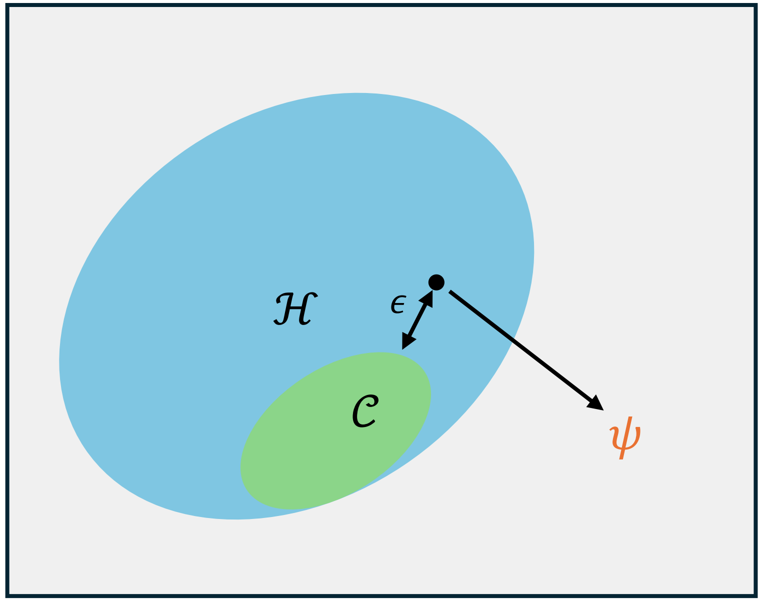

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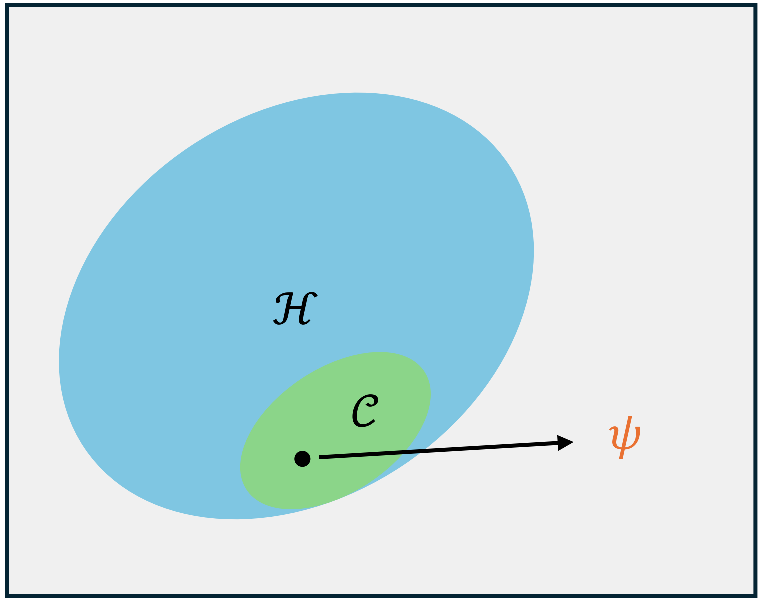

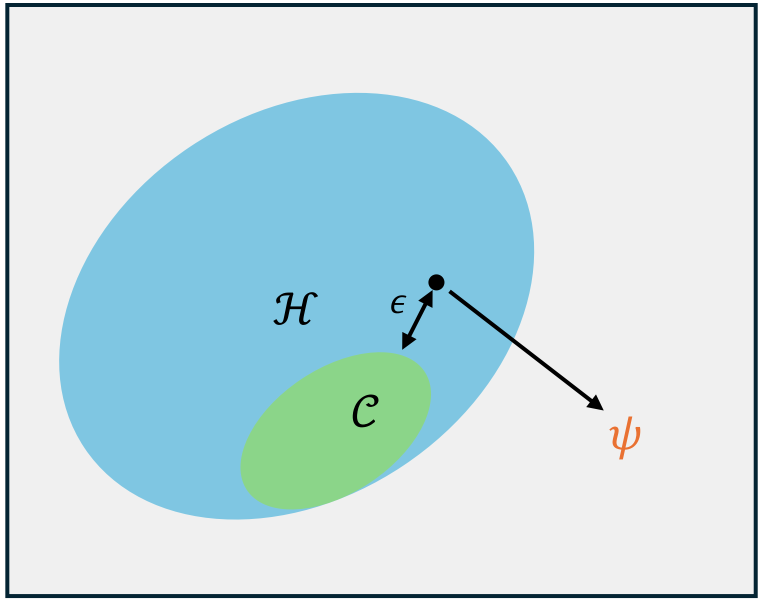

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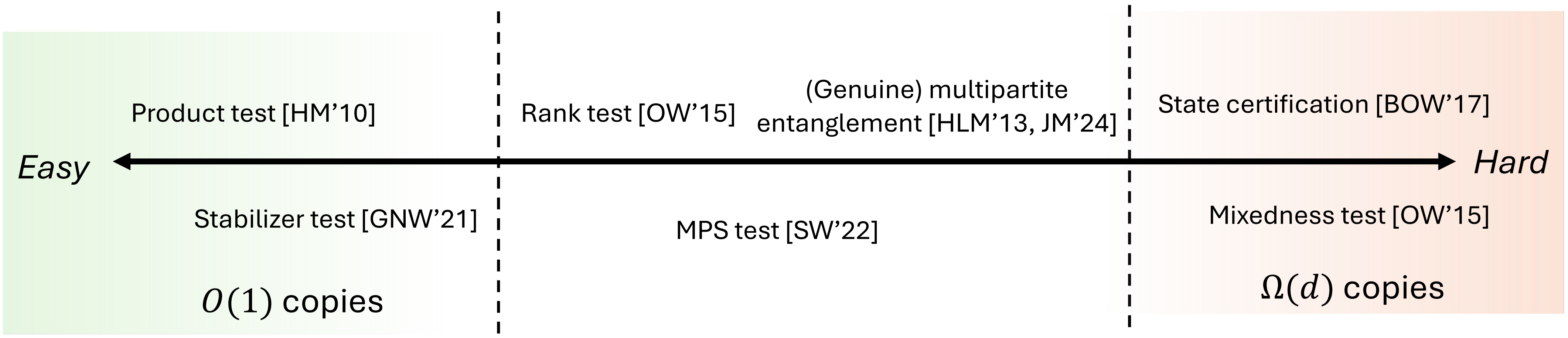
- "Perfect completeness" if $a = 1$

Selected Prior Work



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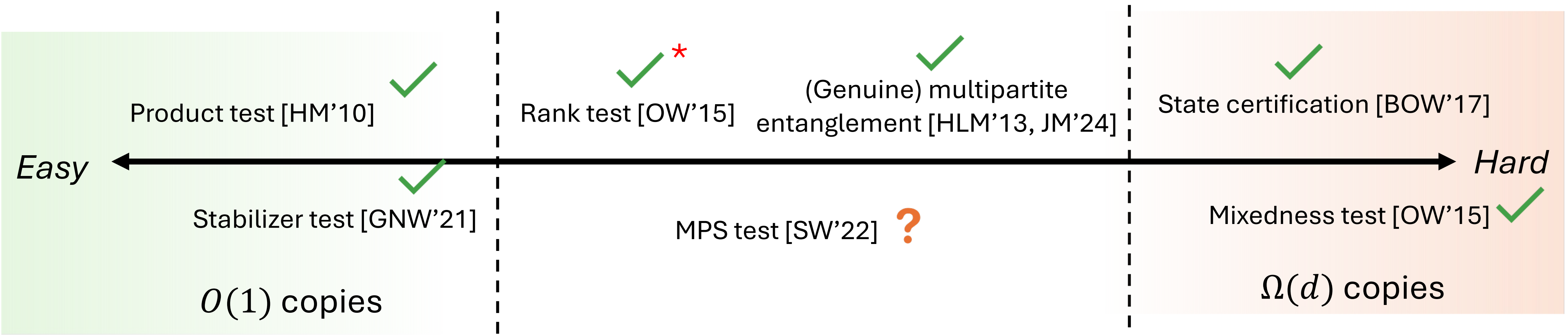

✓ = solved (up to $\log \log d$ factors)

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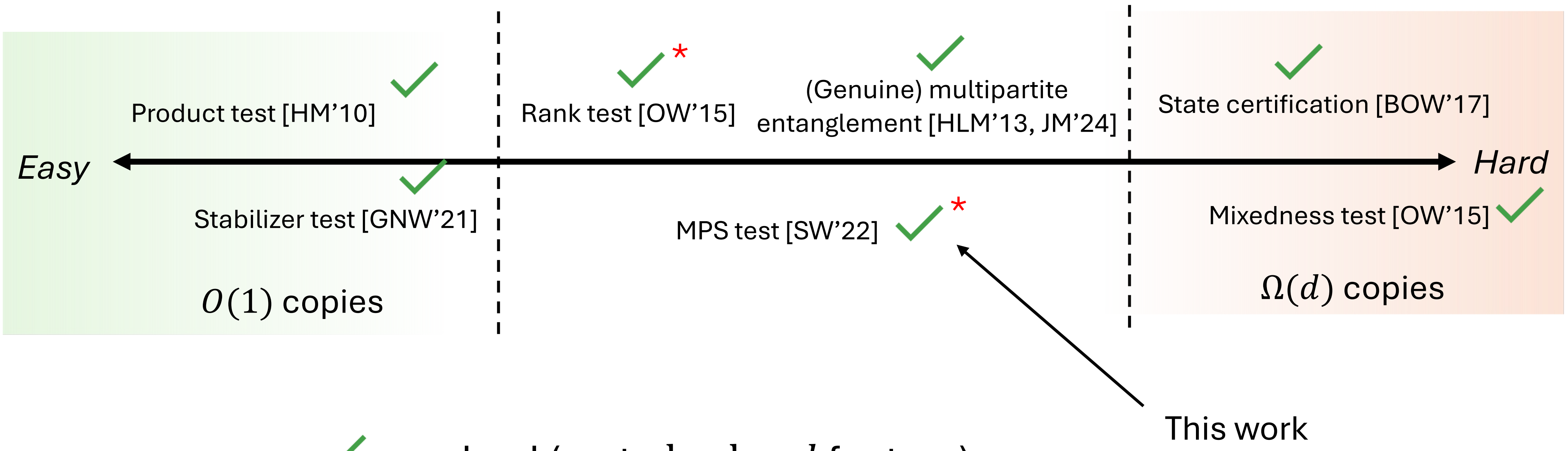

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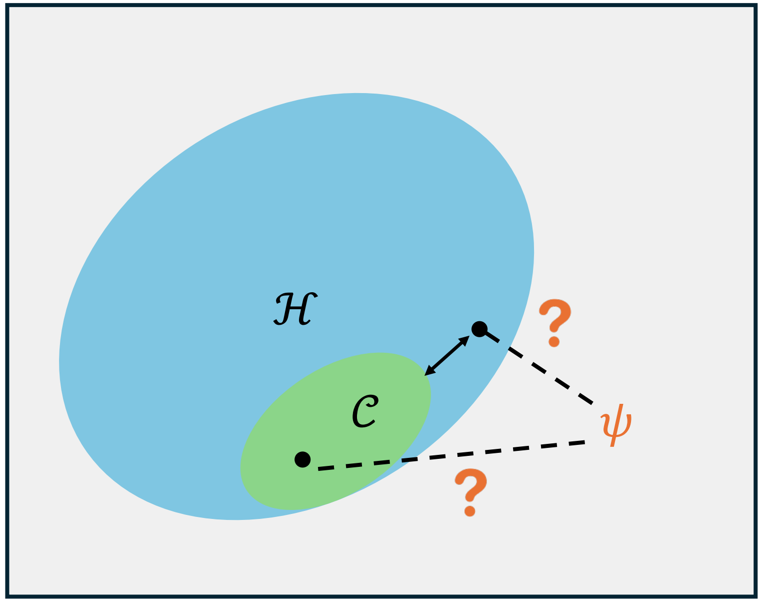

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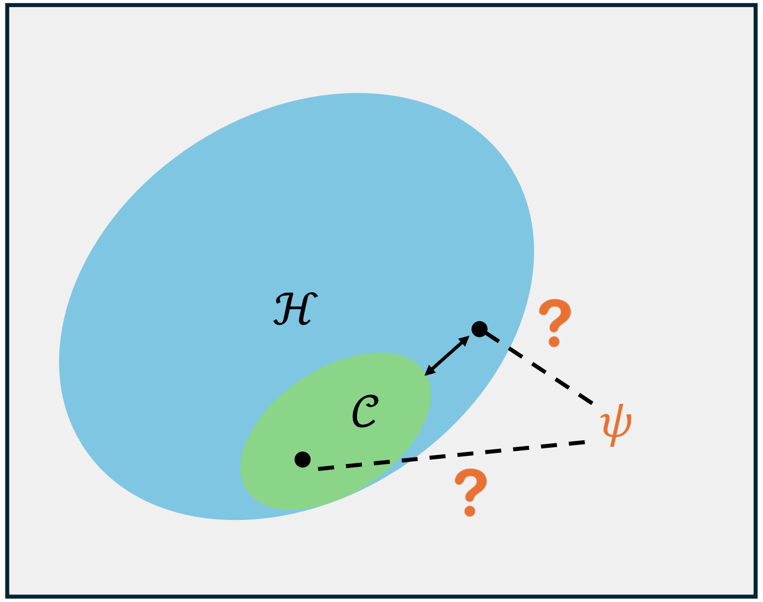
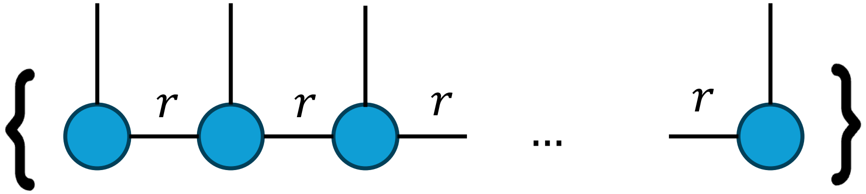


MPS testing:

$\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$

$\mathcal{C} = MPS(r) =$ 

$= \left\{\psi \in \mathcal{H}\colon \psi_{i_1 i_2 \dots i_n} = M^{[1]i_1}_{\alpha_1} M^{[2]i_2}_{\alpha_1\alpha_2} \dots M^{[n]i_{n-1}}_{\alpha_{n-2}\alpha_{n-1}} M^{[1]i_n}_{\alpha_n}\right\}$

$= \{\psi \in \mathcal{H}\colon SR(\psi) \leq r \ \ \forall e \in E\}$

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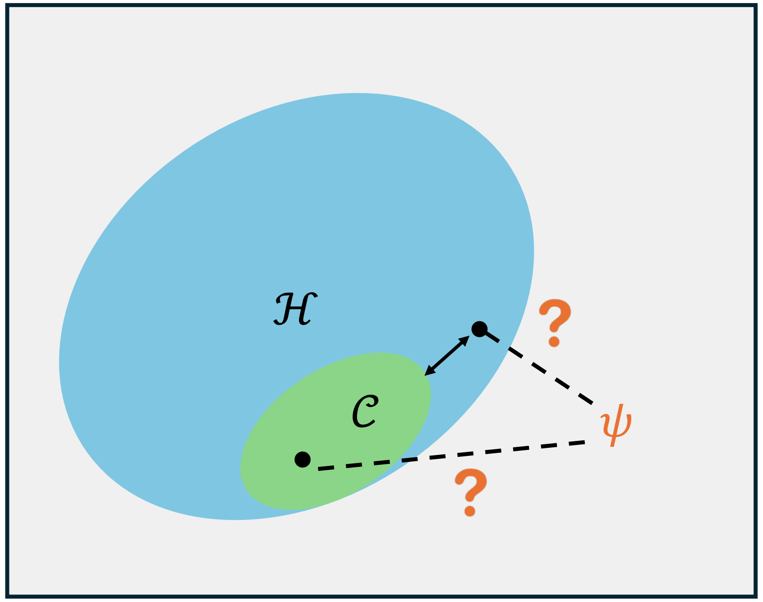

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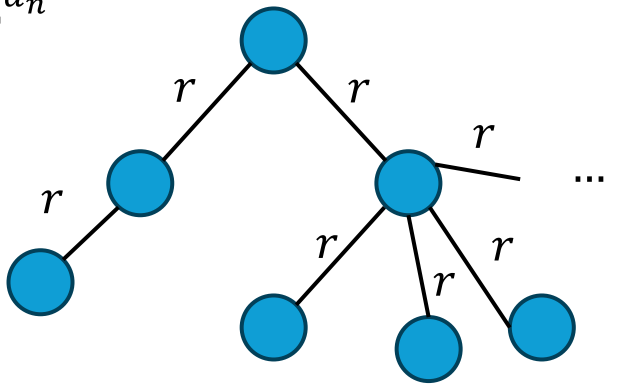

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TTNS testing:

Fix any tree graph $G = (V, E)$ with n vertices.

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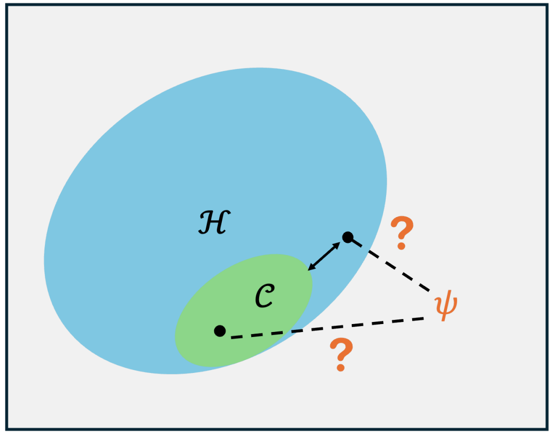

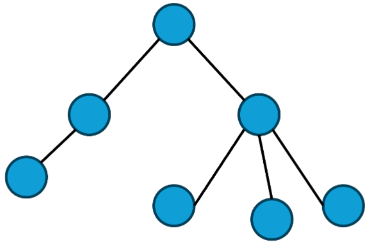

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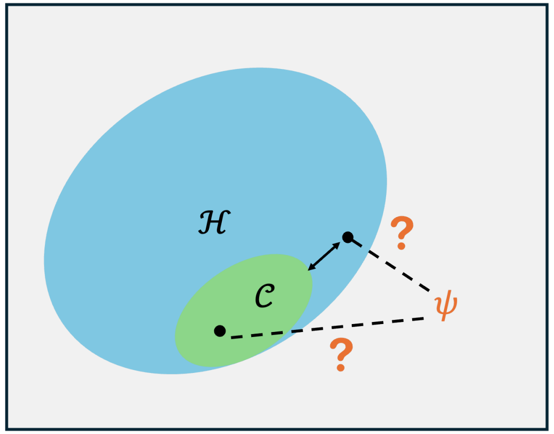

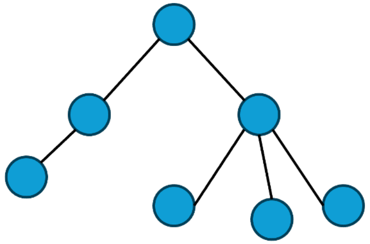

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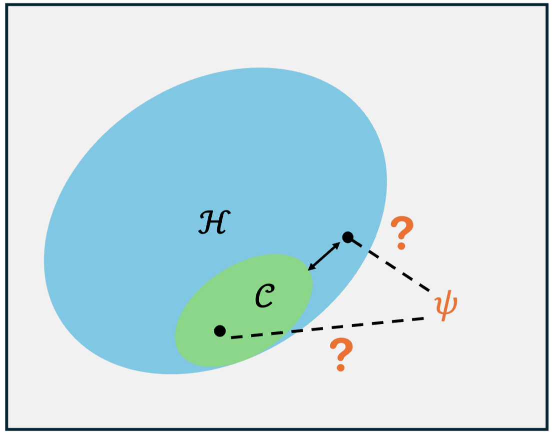

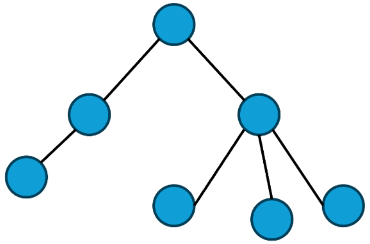

Brief History

- At $r = 1$ this is product testing: $\Theta(1)$ copies necessary and sufficient **[HM10]**
- For MPS at $r \geq 2$: $\Omega(\sqrt{n})$ copies necessary, and $O(nr^2)$ sufficient **[SW22]**
- For MPS at $r \leq 2^{n/8}$: $\Omega(\sqrt{r})$ copies necessary **[Aar+23]**
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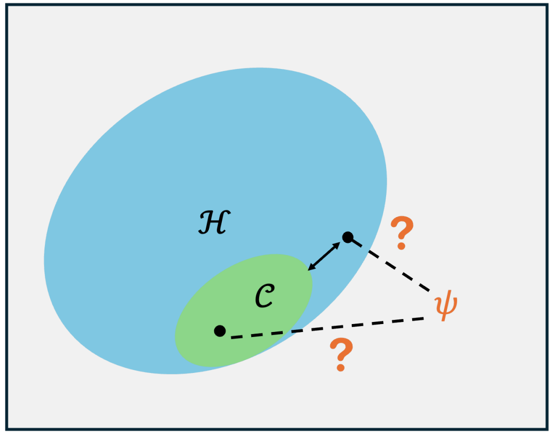

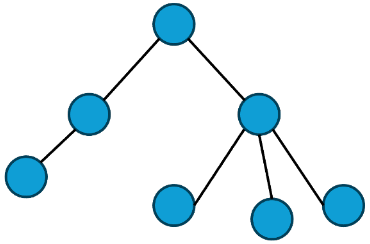

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- For any TTNS at $r \geq 2 + \log(n)$: $\Omega(nr^2/\log n)^*$ copies necessary, and $O(nr^2)$ copies sufficient **[This work]**

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Apply the rank test to every edge [SW'22]:

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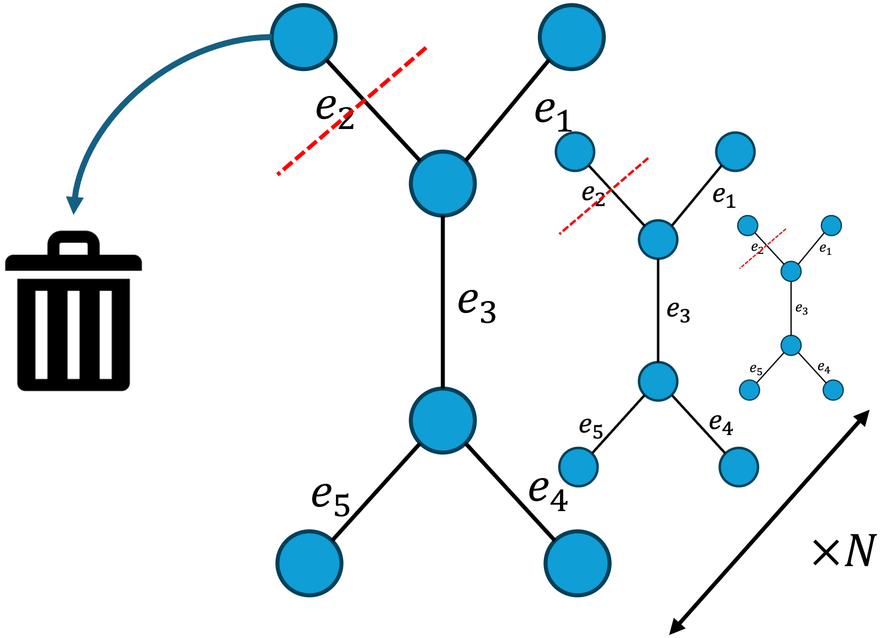

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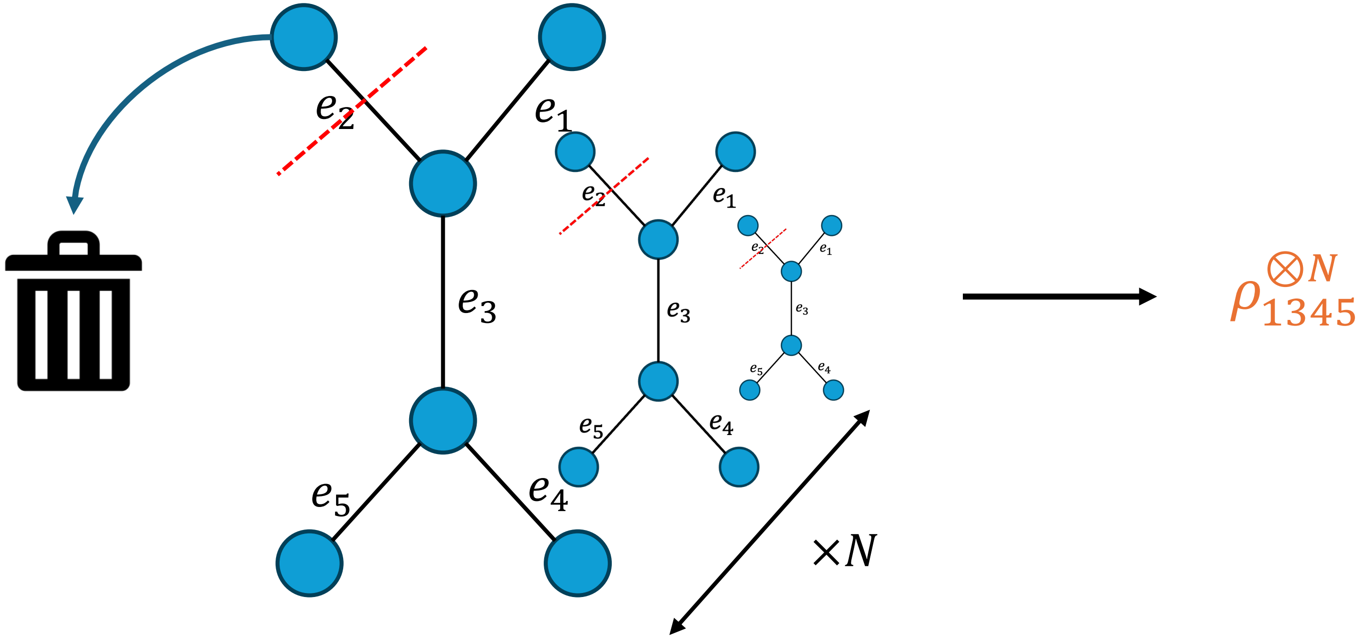

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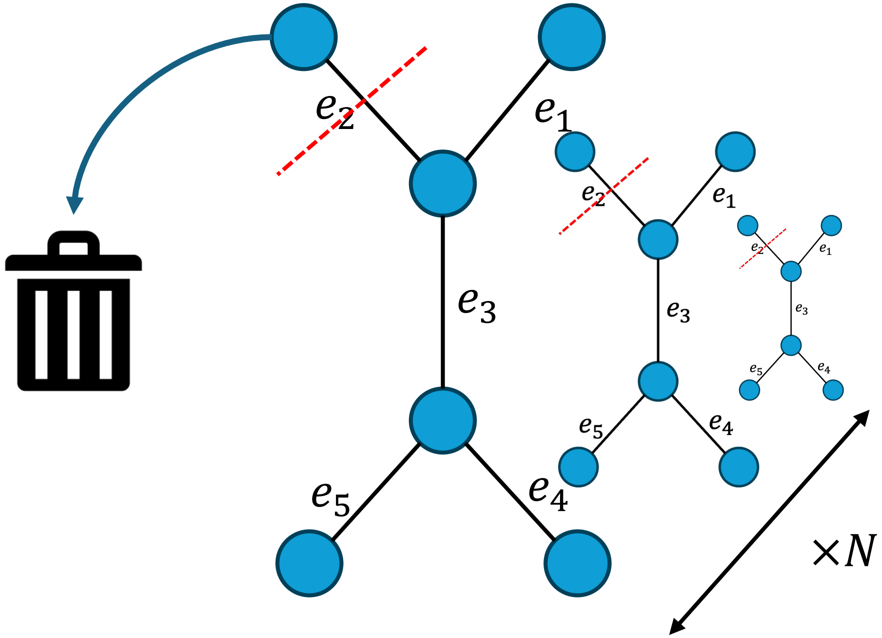

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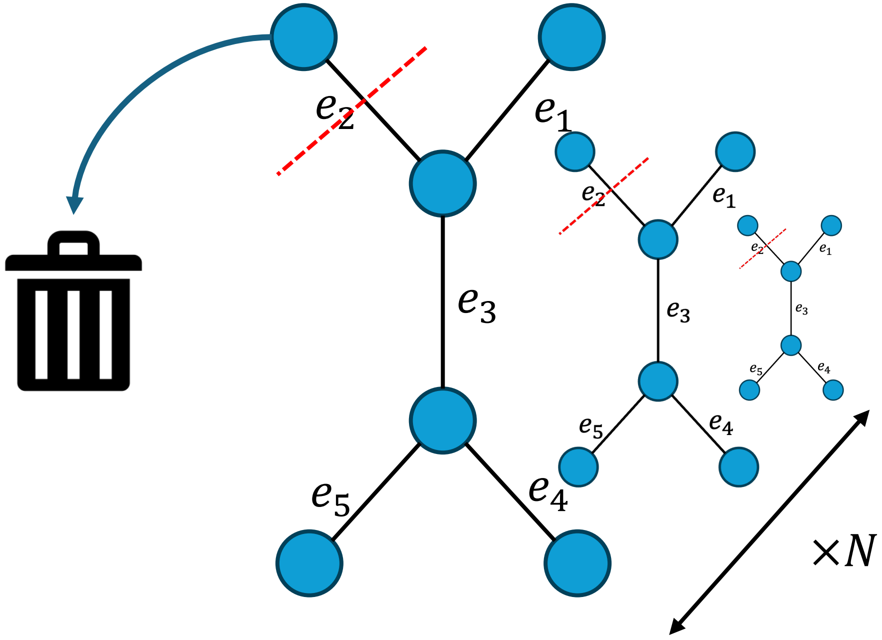

$\longrightarrow$ $\rho_{1345}^{\otimes N}$ $\longrightarrow$ Is $\text{rank}(\rho) \leq r$?

$N = O(nr^2)$ copies suffice to detect far-away state.

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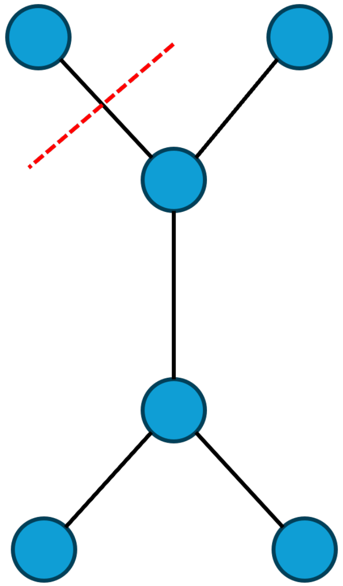

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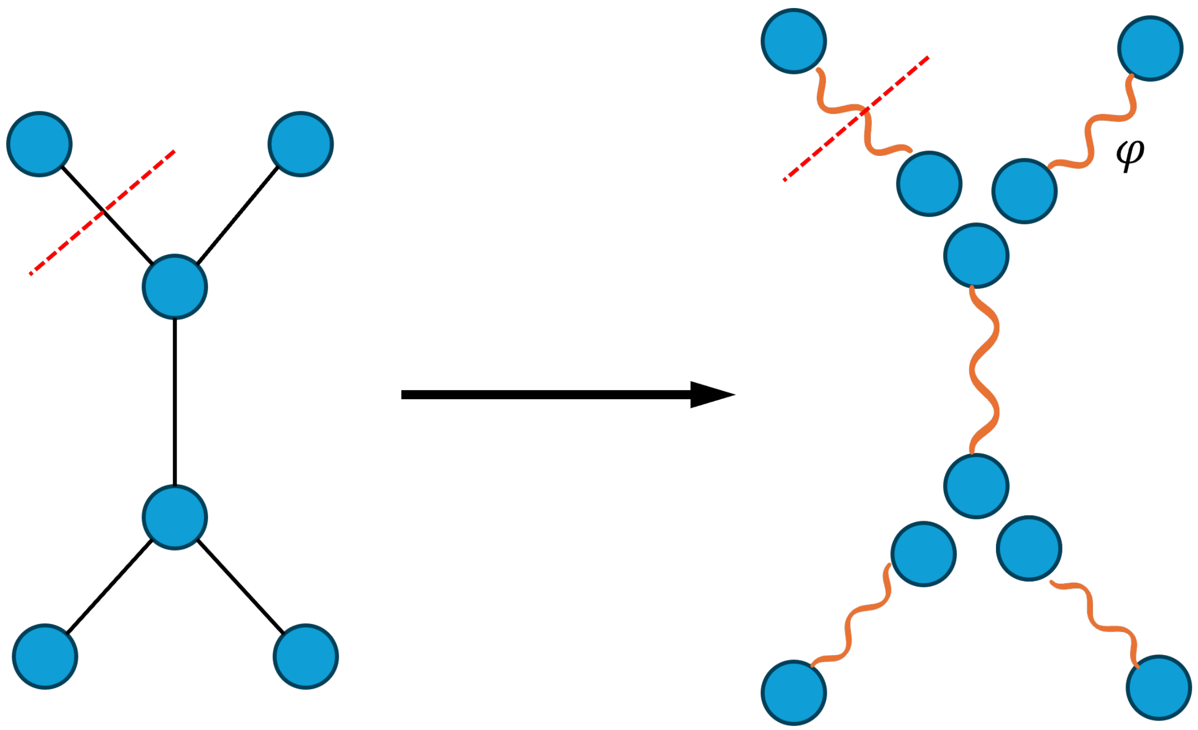

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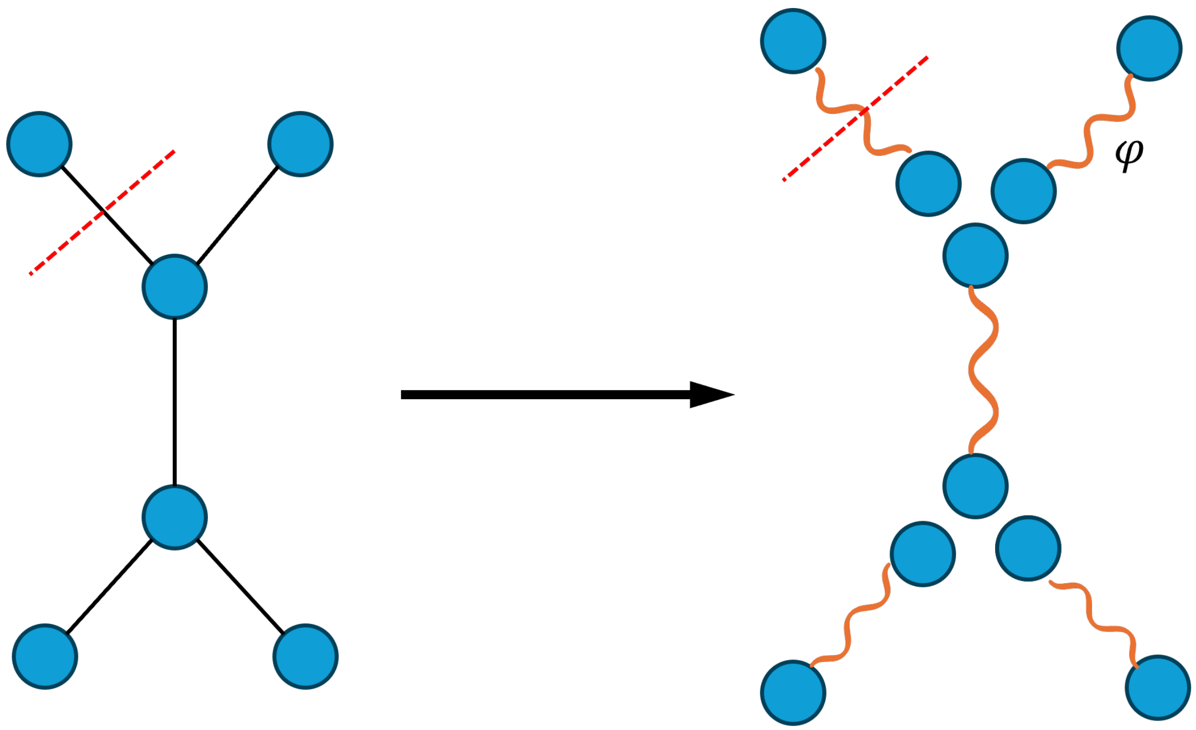

$$Schmidt(\varphi) = \left(\sqrt{1-\frac{\epsilon}{n}}, \sqrt{\frac{\epsilon}{n(d-1)}}, \sqrt{\frac{\epsilon}{n(d-1)}}, \dots, \sqrt{\frac{\epsilon}{n(d-1)}}\right)$$

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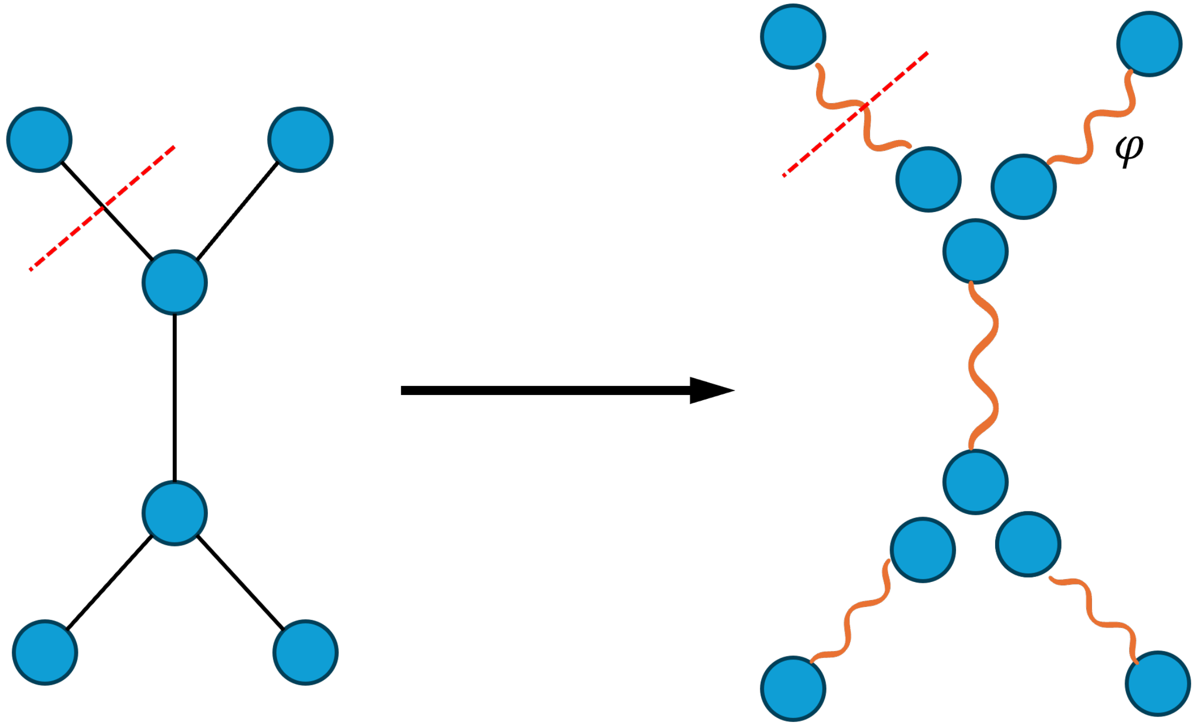

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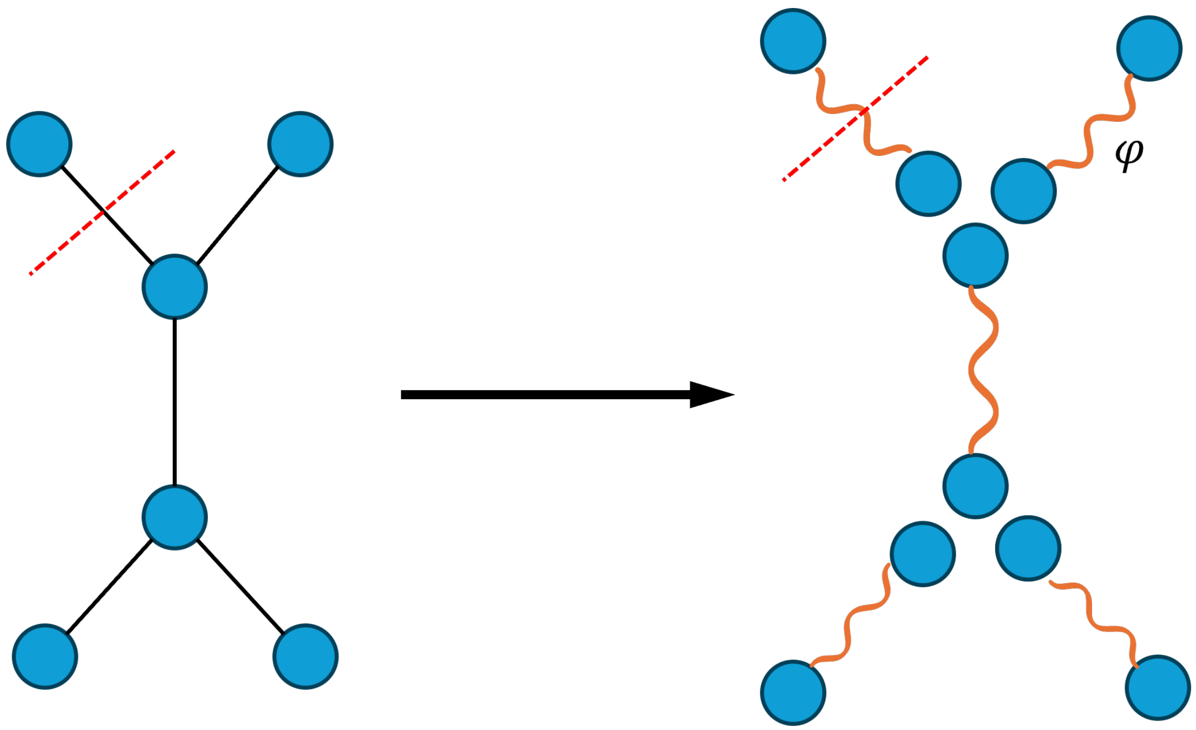

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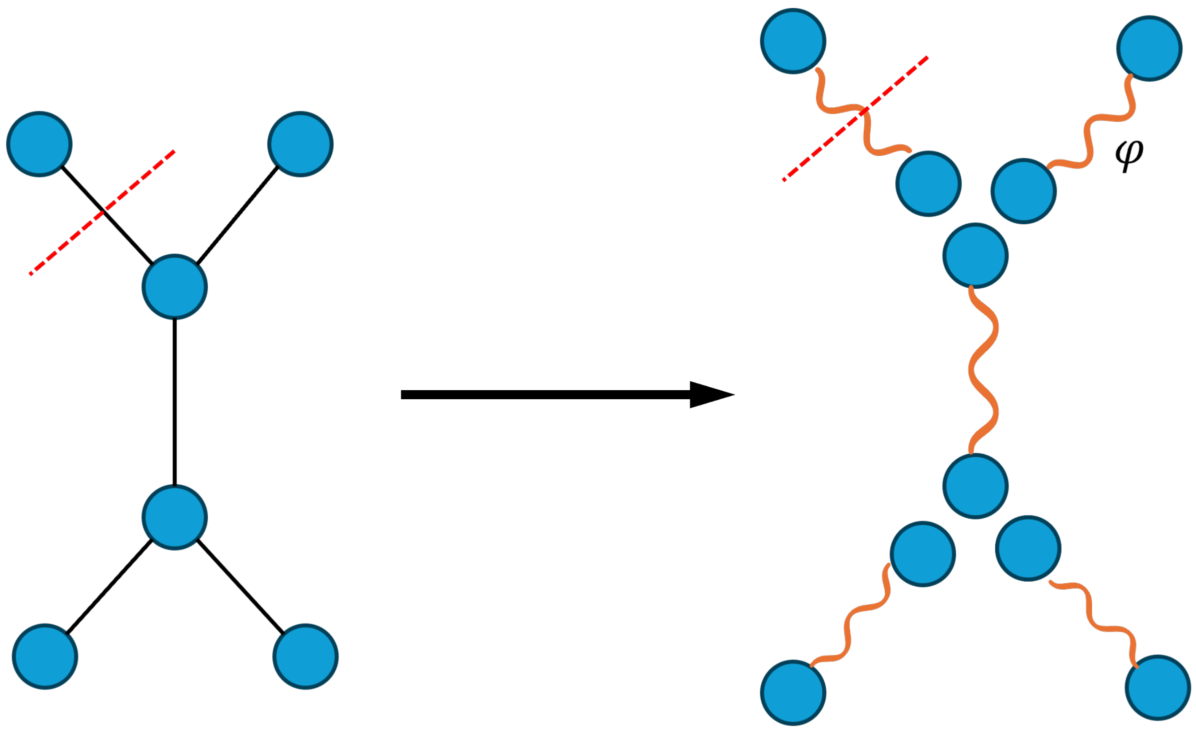

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$\Longrightarrow$ the rank test struggles to reject!

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Fact from representation theory:

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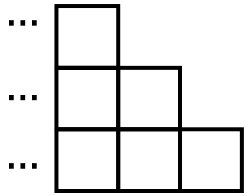

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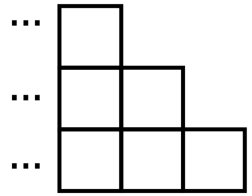

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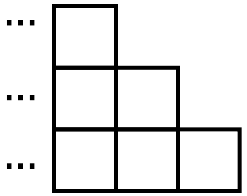

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Schur polynomial

$$\text{Distributed as } p(\lambda) = \mathrm{Tr}\left(I_{\mathcal{P}_\lambda}\right)\mathrm{Tr}\left(q_\lambda^d(\rho)\right) = \dim(\mathcal{P}_\lambda)s_\lambda(p_1, \ldots, p_d)$$

Spectrum of ρ

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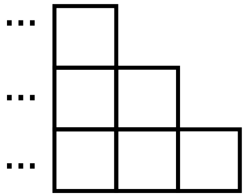

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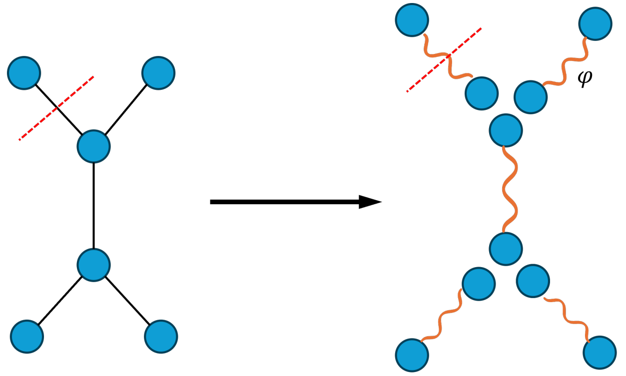
Theorem [OW'15]: *Let $p = (p_1, \dots, p_d)$ be the spectrum of ρ and let $X_1, \dots, X_N \sim p$, iid. It holds that*

$$\mathrm{p_{acc}} \equiv \sum_{\lambda \vdash N : \ell(\lambda) \leq r} \mathrm{Tr}\left(\Pi_\lambda \rho^{\otimes N}\right) = \mathrm{Pr}[LDS(X) \leq r]$$

Lower Bound Proof Sketch



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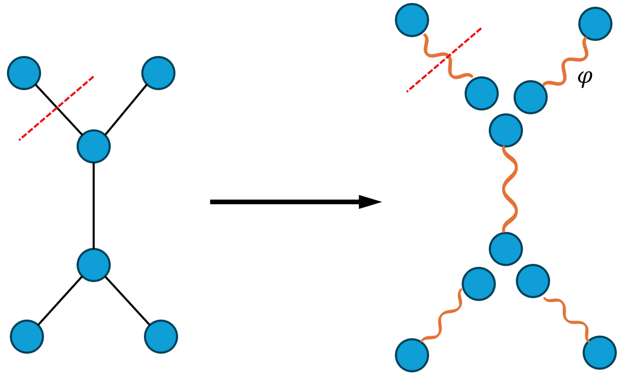
The TTNS tester performs the rank test with respect to every cut, so acceptance probability is $(\mathrm{p}_{acc})^{n-1}$.



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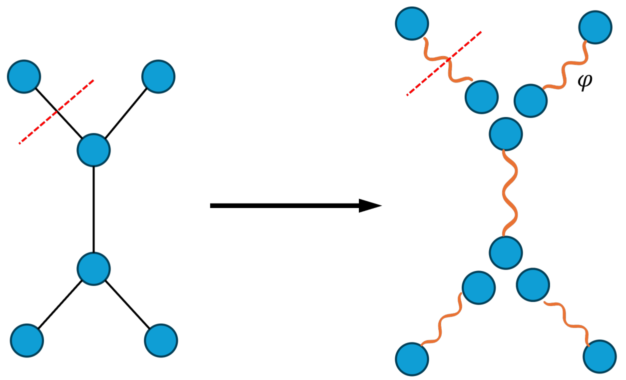

$$Schmidt(\varphi) = \left(\sqrt{1-\frac{\epsilon}{n}}, \sqrt{\frac{\epsilon}{n(d-1)}}, \sqrt{\frac{\epsilon}{n(d-1)}}, \dots, \sqrt{\frac{\epsilon}{n(d-1)}}\right)$$

The TTNS tester performs the rank test with respect to every cut, so acceptance probability is $(\mathrm{p}_{acc})^{n-1}$.

Recall: Rank test accepts with probability

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where X is random word with letters sampled according to $spec(\rho)$.

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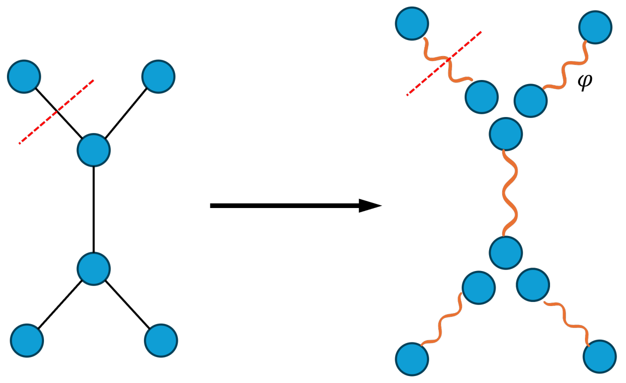

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We get an N-letter word like $X = 1111811111113511121 \ldots$

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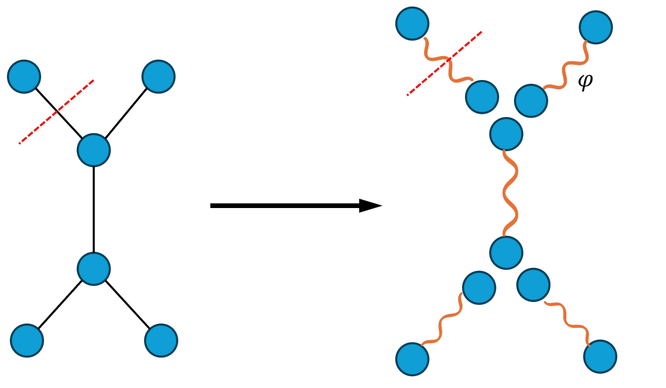

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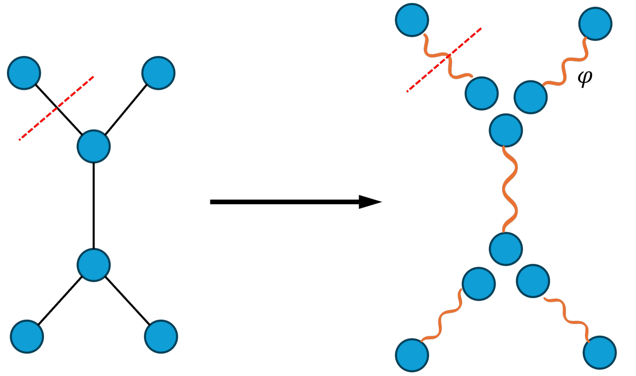

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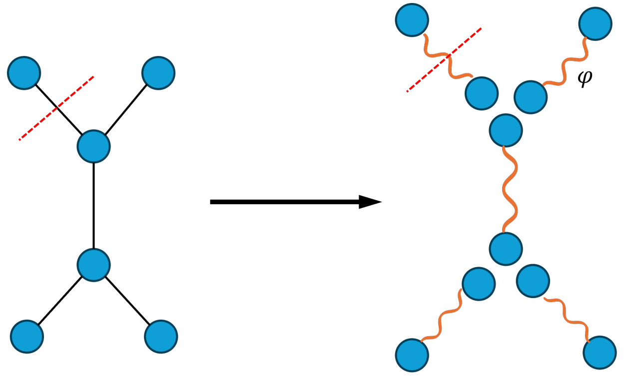

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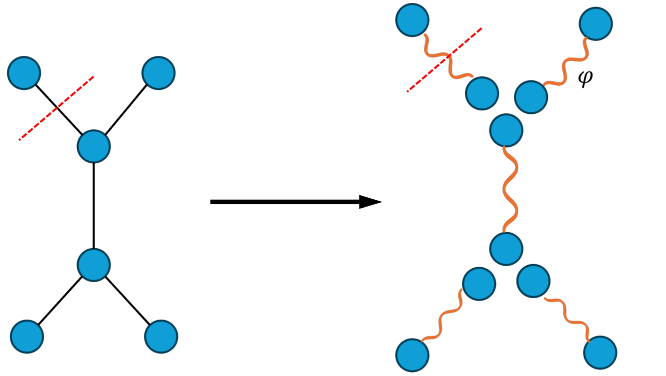

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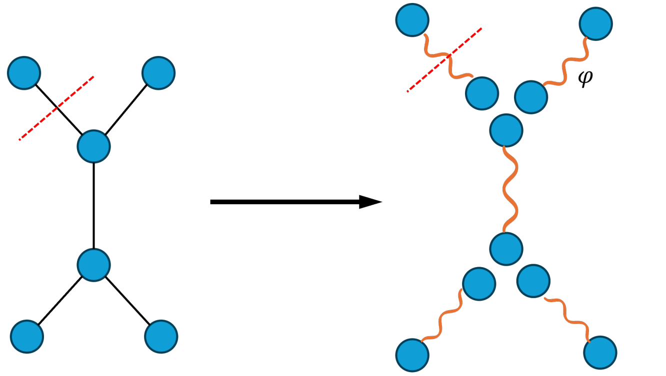

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So if $r \approx \log n$ then $(\mathrm{p}_{acc})^{n-1} \approx \left(1 - \left(\frac{1}{\log n}\right)^{\log n}\right)^n \to 1.$

I. Background & Results
II. Lower Bound for a Specific Algorithm
III. There is No Better Algorithm
IV. Conclusion

Lemma: *The optimal test with perfect completeness for any class $\mathcal{C}$ of* pure *states is*

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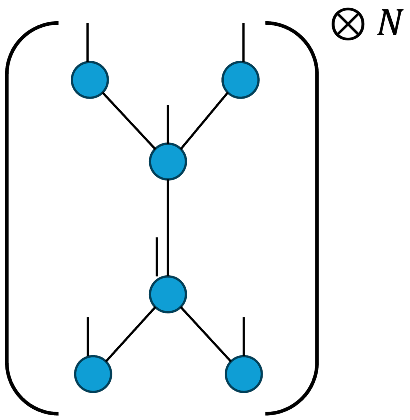
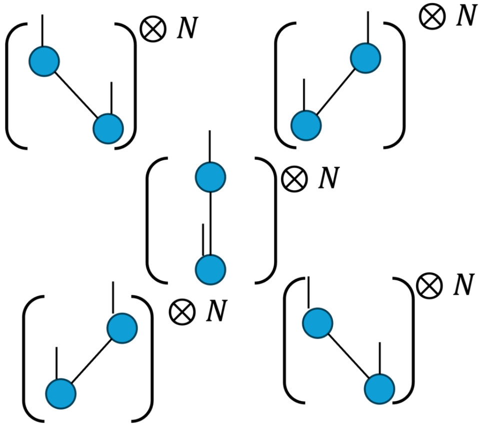
$$\text{proj span}\left\{|\phi\rangle^{\otimes N} : |\phi\rangle \in TTNS(G,r)\right\} = \text{proj span}$$



$$\succcurlyeq \text{proj span}$$



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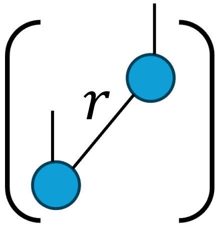
proj span 

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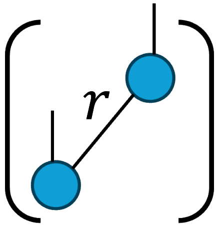
proj span 

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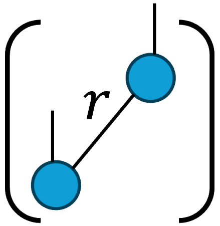

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In general, to test properties of bipartite entanglement, might as well discard Bob's system [MH07, CWZ24, Har05].

I. Background & Results
II. Lower Bound for a Specific Algorithm
III. There is No Better Algorithm
IV. Final Remarks

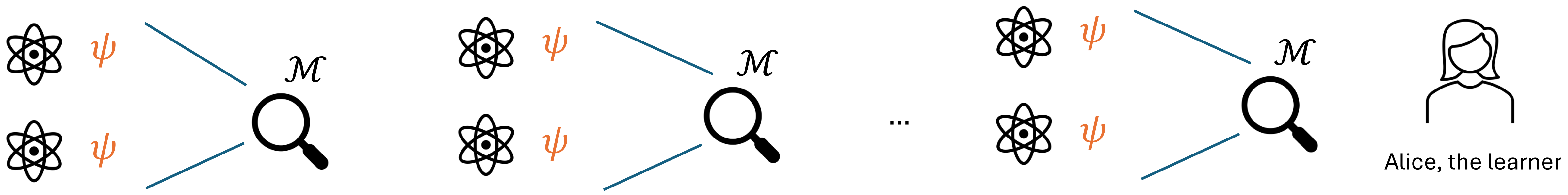
Conclusion

- We give nearly tight bounds for testing MPS/TTNS with one-sided error when bond dimension grows logarithmically.
- We also analyze few-copy tests (not discussed here)
- Open questions:
  - What happens at constant bond dimension? We suspect $O(\sqrt{n})$ copies could suffice. (See bonus slides.)
  - What about two-sided error? (We don't even know the answer for rank testing.)
  - What is the copy complexity of *learning* MPS?
  - What is the copy complexity of learning/testing PEPS, or MPS with CBC?

Bonus slides

Few-copy Tests

- Besides copy complexity, an important resource is quantum memory, i.e., number of copies measured simultaneously
- For the product test (MPS/TTNS testing at $r = 1$), only two copies at a time:

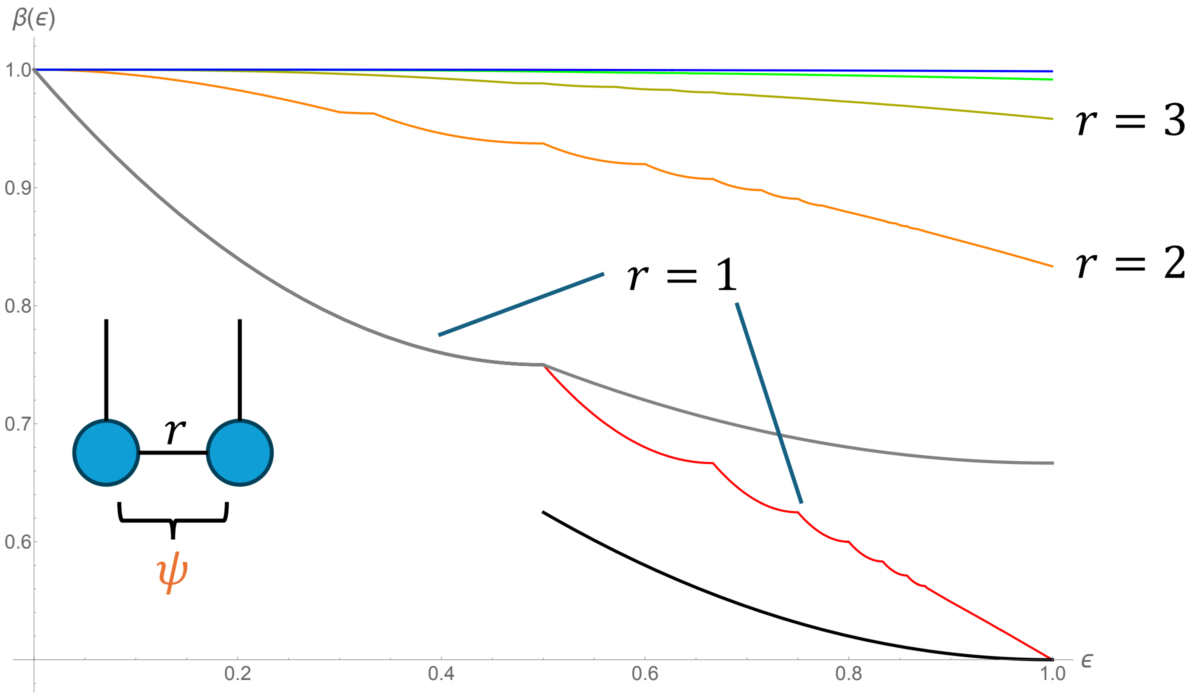


Q: What about $r \geq 2$?

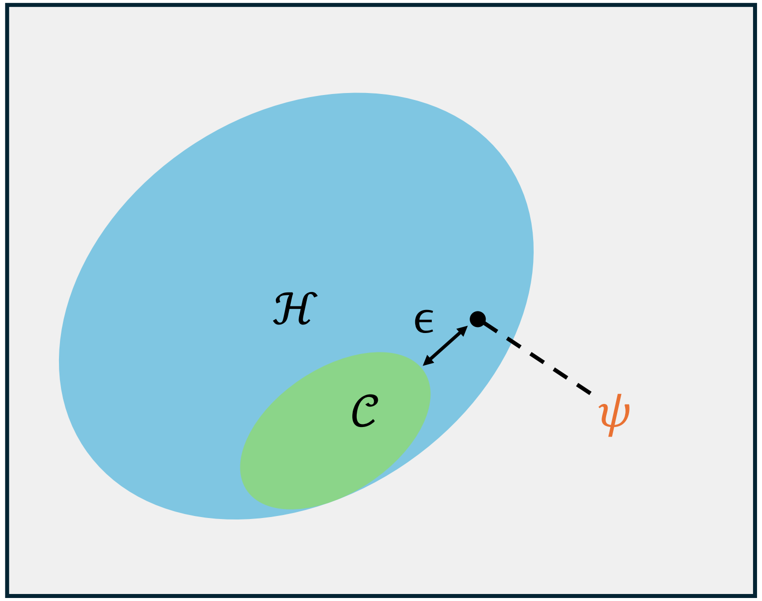
Few-copy Tests

Theorem: *For any $r \geq 2$, consider testing $\psi \in TTNS(G, r)$ with measurements on $(r+1)$ copies at a time (w/ one-sided error). It holds that $O(n^r)$ total copies suffice and $\Omega(n^{r-1})$ copies are necessary.*

$n = 2$, optimal Schmidt-rank test



$\beta(\epsilon)$ = acceptance probability on ϵ-far state



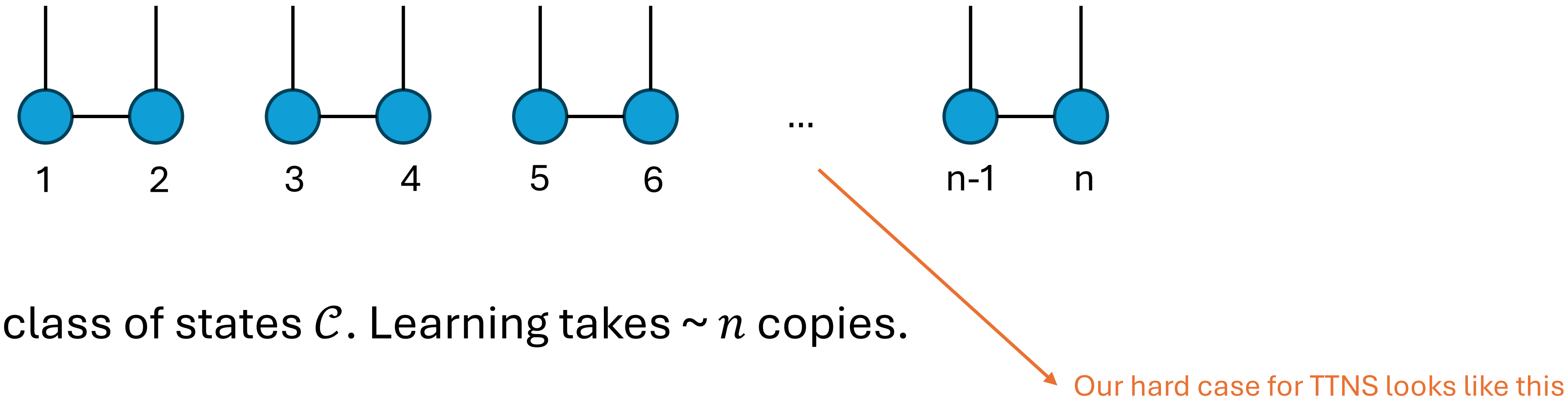
Constant Bond Dimension?

- For any TTNS at $r \geq 2 + \log(n)$: $\Omega(nr^2/\log n)^*$ copies necessary, and $O(nr^2)$ copies sufficient **[This work]**

* *with perfect completeness*

Q: Why do we need $r \geq 2 + log\ n$?

Take $r = 2$ and "forget" half the bonds:



This is a valid class of states $\mathcal{C}$. Learning takes $\sim n$ copies.

This work: testing $\mathcal{C}$ possible using just $O(\sqrt{n})$ copies.