Random dimension reduction

& learning symmetric properties of quantum states

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arxiv:2606.23592

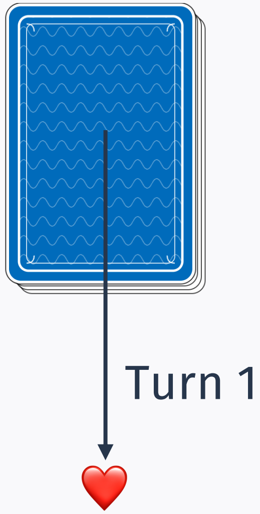
Results

We study a quantum channel that simultaneously reduces the dimensions of multiple input states while preserving certain symmetric properties of the input.

This leads to improved upper bounds for several quantum learning tasks.

This also completes the story for optimal quantum tomography without representation theory.

Learning symmetric properties



$\boxed{1}$

♣ ♥ ♠ ♦

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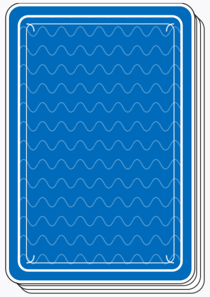

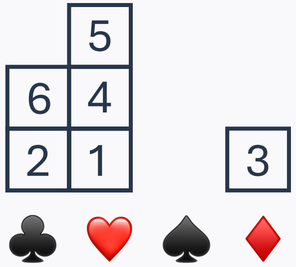

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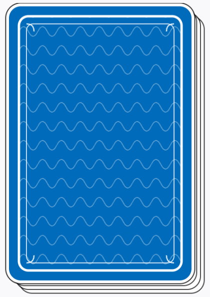

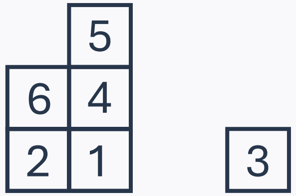

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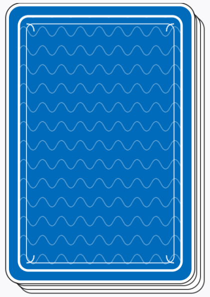

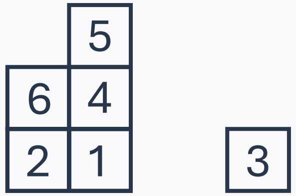

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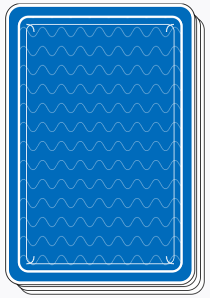

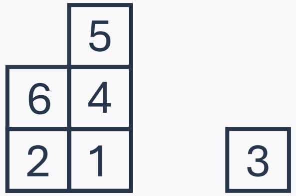

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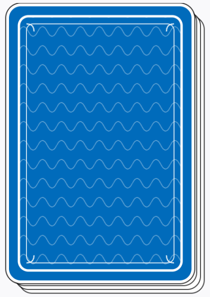

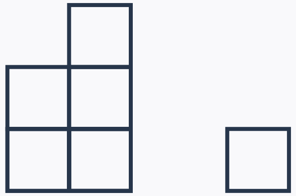

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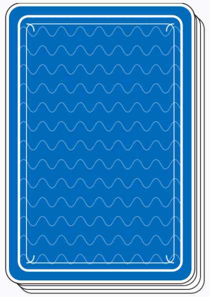

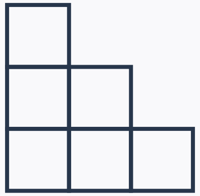

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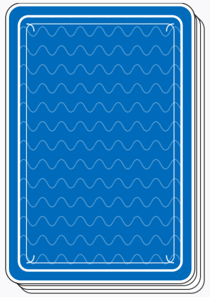

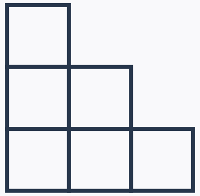

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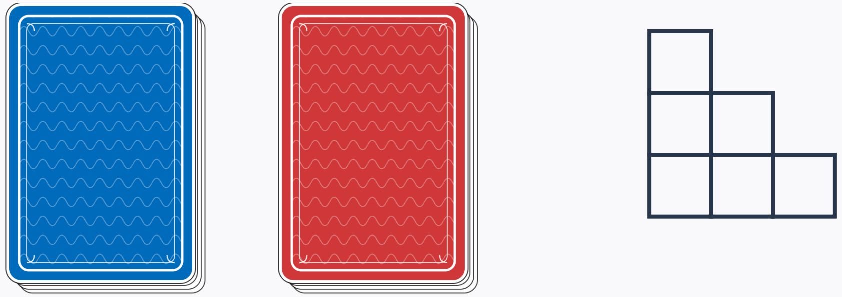

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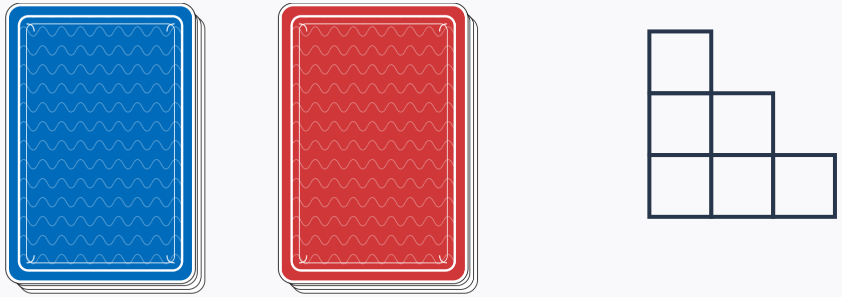

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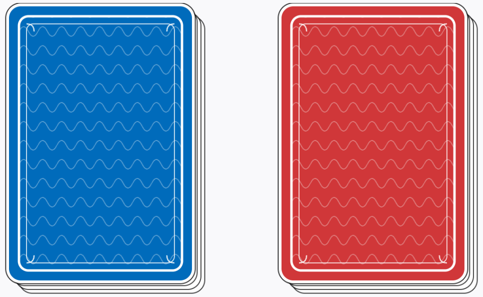

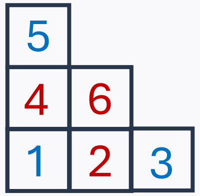

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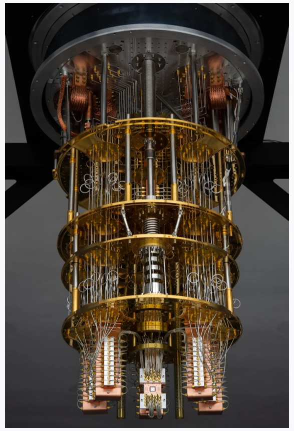
VV'11 – Valiant & Valiant. Estimating the unseen... (STOC'11)
WY'16 – Wu & Yang. Chebyshev polynomials... (Ann. Statist. 47(2))

Symmetric quantum properties



$$\longrightarrow \rho^{\otimes n}$$

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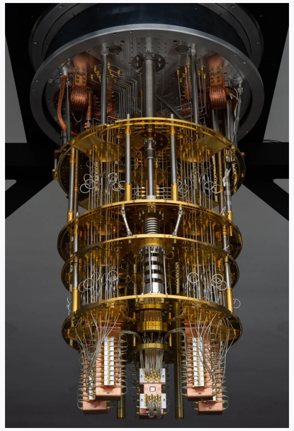

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Problem (Entropy estimation)

Given: n copies of $\rho \in \mathbb{C}^{d\times d}$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0$

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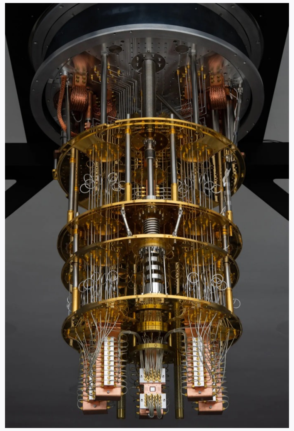

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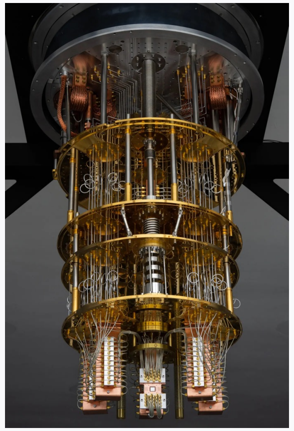

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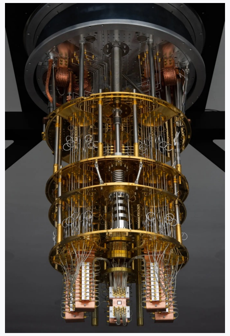

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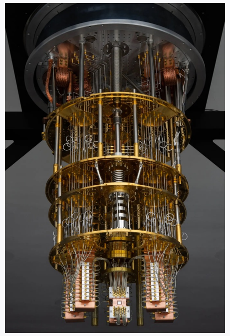

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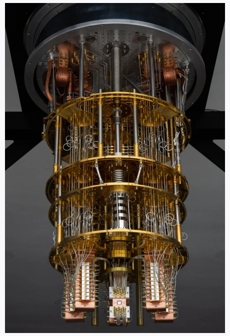

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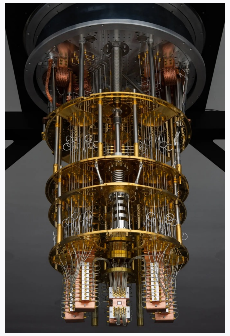

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Applications: *verification [Elb+'20], probing entanglement [SW'22,LL'26], quantum algorithms [CHW'07]*

[Elb+'20] - Elben et al. Cross-platform verification... (PRL 124(1))

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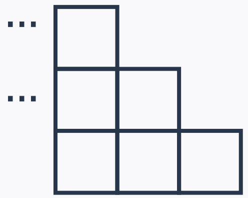

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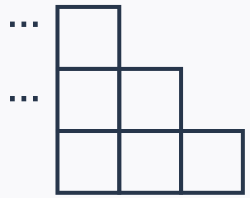

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Step 2: Empirical Young diagram [KW'01] $\longrightarrow$ $O(r^2)$ copies suffice[1] $\boxed{\textit{independent of } d}$

[1]Recently (last week) improved to $O\left(r^2 \frac{\log\log r}{\log r}\right)$ [PSTW'26]

[PSTW'26] – Pelecanos, Spilecki, Tang, Wright. The Keyl-Werner algorithm is not optimal... (2026)

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Theorem (Random purification channel [TWZ'25])

$\forall r\ \exists$ a channel $\mathcal{P}^{(r)}$ with the action

$$\mathcal{P}^{(r)}\left(\rho^{\otimes n}\right) = \mathbb{E}_{\phi^\rho} |\phi^\rho\rangle\langle\phi^\rho|^{\otimes n}$$

for any quantum state ρ of rank at most r where $|\phi^\rho\rangle \in \mathbb{C}^d \otimes \mathbb{C}^r$ is a random purification, satisfying $\mathrm{tr}_2(|\phi^\rho\rangle\langle\phi^\rho|) = \rho$

If $\rho = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\psi_i|$,

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$$\rho^{\otimes n} \xrightarrow{\ \mathcal{P}^{(r)}\ } |\phi^\rho\rangle\langle\phi^\rho|^{\otimes n} \xrightarrow{\ \mathrm{tr}_1\ } \rho_0^{\otimes n} \qquad \rho_0 \in \mathbb{C}^{r\times r}, \quad \mathrm{spec}(\rho) = \mathrm{spec}(\rho_0)$$

Recall: Full state tomography has sample complexity $O(\dim(\rho_0)^2)$ [HHJWY'16, OW'16]

[HHJWY'16] – Haah et al. Sample-optimal quantum state tomography. (STOC'16)

[OW'16] – O'Donnell & Wright. Efficient quantum tomography. (STOC'16)

[TWZ'25] – Tang, Wright, Zhandry. Conjugate queries can help. (2025)

Limitations of random purification

Problem (Fidelity estimation)

Given: n copies of $\rho \in \mathbb{C}^{d\times d}$ and n copies of $\sigma \in \mathbb{C}^{d\times d}$

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If ρ and σ both have rank at most r, then $\widetilde{\Theta}(r^2)$ copies suffice [UNWT'25]

[UNWT'25] – Utsumi, Nakata, Wang, Takagi. Quantum algorithms for the Uhlmann transformation. (2025)

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Q: Can we see this again with the random purification channel?

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Problem: both approaches throw away information about the eigenbases of ρ and σ

Dimension reduction: key idea

Pick a different channel $\mathcal{R}$ with the action

$$\begin{tikzcd}
\rho^{\otimes n} \arrow[r, "\mathcal{P}^{(r)}"] \arrow[rr, bend right, "\mathcal{R}"'] & |\phi^\rho\rangle\langle\phi^\rho|^{\otimes n} \arrow[r, "\mathrm{tr}_1"] & \rho_0^{\otimes n}
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Dimension reduction: key idea

Pick a different channel $\mathcal{R}$ with the action

$$\begin{tikzcd} \rho^{\otimes n} \arrow[r, "\mathcal{P}^{(r)}"] \arrow[rr, bend right, "\mathcal{R}"'] & |\phi^\rho\rangle\langle\phi^\rho|^{\otimes n} \arrow[r, "\mathrm{tr}_1"] & \rho_0^{\otimes n} \end{tikzcd} \qquad \rho_0 \in \mathbb{C}^{r\times r}, \quad \mathrm{spec}(\rho) = \mathrm{spec}(\rho_0)$$

Then hopefully we also have

$$(\rho \otimes \sigma)^{\otimes n} \xrightarrow{\mathcal{R}} (\rho_0 \otimes \sigma_0)^{\otimes n}$$

where ρ_0, σ_0 have rank $O(r)$, *and* contain information about the eigenbases of ρ, σ

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Definition (Dimension reduction map)

$$\mathcal{R} : \mathrm{L}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right) \to \mathrm{L}\left(\left(\mathbb{C}^k\right)^{\otimes n}\right)$$

$$\mathcal{R}(X) = \mathbb{E}\left[\left(\boldsymbol{W}^\dagger\right)^{\otimes n} M^{-1/2} X M^{-1/2} \boldsymbol{W}^{\otimes n}\right]$$

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1) $X = \rho^{\otimes n}$ for some rank-r ρ. Set $k = r$.

$$\mathcal{R}(\rho^{\otimes n}) = \mathop{\mathbb{E}}_{\boldsymbol{U}\sim\mathbb{U}(r)} (\boldsymbol{U}\rho_0\boldsymbol{U}^\dagger)^{\otimes n}, \qquad \mathrm{spec}(\rho_0) = \mathrm{spec}(\rho)$$

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2) $X = \rho^{\otimes n} \otimes \sigma^{\otimes n}$ for some ρ, σ with ranks r_1, r_2 respectively. Set $k = r_1 + r_2$, $n \to 2n$

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where $\rho = W\rho_0 W^\dagger$ and $\sigma = W\sigma_0 W^\dagger$ for an isometry $W : \mathbb{C}^k \to \mathbb{C}^d$

Application to learning symmetric properties

A straightforward “baseline” strategy: **random dimension reduction + tomography**

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For ρ, σ with ranks at most r,

$$\rho^{\otimes n} \otimes \sigma^{\otimes n} \xrightarrow{\mathcal{R}} \rho_0^{\otimes n} \otimes \sigma_0^{\otimes n} \xrightarrow{\text{tomography}} \hat{\rho}_0, \hat{\sigma}_0 \longrightarrow \text{estimate properties}$$

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If $f(\rho, \sigma)$ is symmetric and continuous, so that $f(\hat{\rho}_0, \hat{\sigma}_0) \approx f(\rho, \sigma)$, then $n = O(r^2)$ copies suffice.

Application to learning symmetric properties

Estimation task	Best prior	**This work**
Trace distance	$O(r^2 \cdot \log^2(1/\varepsilon)/\varepsilon^4)$ [WZ24a; CWZ25]	$O(r^2/\varepsilon^2)$
Fidelity	$O(r^2 \cdot \log^2(1/\varepsilon)/\varepsilon^4)^*$ [UNWT25]	$O(r^2/\varepsilon^2)$
α-Tsallis relative entropy, $0 < \alpha < 1/2$	$O(r^{2+2\alpha}/\varepsilon^{2/\alpha+2/(1-\alpha)})$ [CWZ25]	$O(r^2/\varepsilon^{1/\alpha})$
Squared Hellinger distance	$O(r^3 \cdot \log^2(r/\varepsilon)/\varepsilon^8)$ [CWZ25]	$O(r^2/\varepsilon^2)$

Proof sketch

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$$\begin{aligned}\mathcal{R}(X) &= \mathbb{E}\left[\left(\boldsymbol{W}^\dagger\right)^{\otimes n} M^{-1/2} X M^{-1/2} \boldsymbol{W}^{\otimes n}\right] \\ &= \mathop{\mathbb{E}}_{\boldsymbol{V}\sim\mathbb{U}(k)} \boldsymbol{V}^{\otimes n}\left[\left(\boldsymbol{W}^\dagger\right)^{\otimes n} M^{-1/2} X M^{-1/2} \boldsymbol{W}^{\otimes n}\right]\left(\boldsymbol{V}^\dagger\right)^{\otimes n}\end{aligned}$$

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&= \sum_{\pi,\sigma\in S_n} \mathrm{Wg}\,(k,\sigma^{-1}\pi) \mathbb{E}\,\mathrm{tr}\left(F^k_{\sigma^{-1}} \left(\boldsymbol{W}^\dagger\right)^{\otimes n} M^{-1/2} X M^{-1/2} \boldsymbol{W}^{\otimes n}\right) F^k_\pi
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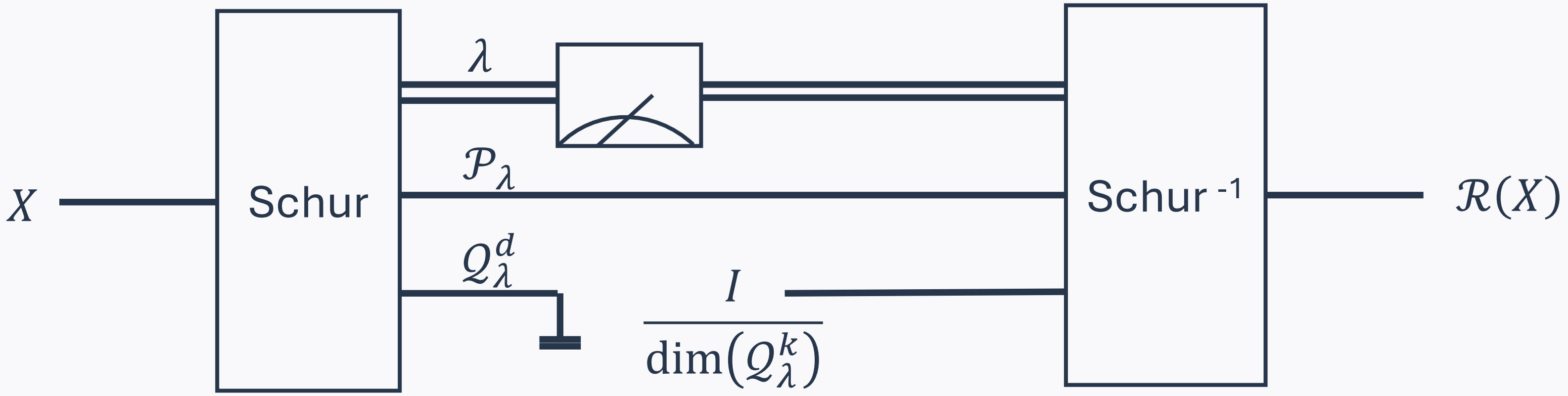
&= \sum_{\pi,\sigma\in S_n} \mathrm{Wg}\,(k,\sigma^{-1}\pi)\,\mathrm{tr}\left(F^d_{\sigma^{-1}} X\right) F^k_\pi
\end{aligned}$$

Efficient implementation

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathcal{P}_\lambda \otimes \mathcal{Q}_\lambda^d$$



See also [YSM'23]

[YSM'23] – Yoshida, Soeda, Murao. Universal construction of decoders from encoding black boxes. (Quantum 7, 2023)

Random purifications revisited

Recall the *Choi-Jamiolkowski isomorphism*

$$J_{\mathcal{N}} = (\mathrm{id} \otimes \mathcal{N})(|\Gamma\rangle\langle\Gamma|), \quad |\Gamma\rangle = \sum_{i=1}^{d} |i\rangle |\, i\rangle$$

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A: the random purification channel, when restricted to *exchangeable* inputs X, e.g., $\rho^{\otimes n}$

Quantum tomography revisited

Optimal quantum tomography with $n = \Theta(dr)$ copies no longer requires representation theory to describe.

$$\rho^{\otimes n} \xrightarrow{\mathcal{P}^{(r)}} |\phi^\rho\rangle\langle\phi^\rho|^{\otimes n} \xrightarrow{\text{Pure tomography}} |\hat{\phi}^\rho\rangle$$

(Observed in [PSTW'26])

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$|\phi^\rho\rangle$ is D-dimensional with $D = dr$.

Pure tomography with $n = O(D)$ copies: just uses church of the symmetric subspace [Har13].

$$\begin{aligned}\int \mathrm{tr}\left(D[n]|u\rangle\langle u|^{\otimes n} \cdot |\psi\rangle\langle\psi|^{\otimes n}\right)|\langle u|\psi\rangle|^2 du &= D[n]\mathrm{tr}\left(\int |u\rangle\langle u|^{\otimes n+1}\, du \cdot |\psi\rangle\langle\psi|^{\otimes n+1}\right) \\ &= \frac{D[n]}{D[n+1]} \\ &\approx 1 - \frac{D}{n}\end{aligned}$$

[PSTW'26] – Pelecanos, Spilecki, Tang, Wright. Mixed state tomography reduces to pure state tomography. (2025)

[Har13] – Aram Harrow. The Church of the symmetric subspace. (2013)

Conclusion

- Random dimension reduction sets baseline for the sample complexity of learning symmetric quantum properties. When can we beat it?
  - First example recently shown in [PSTW'26].
- Our construction also shows that it can be useful to apply the Schur transform to states of the form $\rho^{\otimes a} \otimes \sigma^{\otimes b}$
- Minimum vs maximum rank: for fidelity estimation, natural dependence is on the minimum rank. Can we derive an upper bound of $O\left(r_{\min}^2/\varepsilon^2\right)$?
- Robustness: approximate, not exact, low rank is a more physically realistic assumption.

[PSTW'26] – Pelecanos, Spilecki, Tang, Wright. The Keyl-Werner algorithm is not optimal for spectrum estimation (2026)